Sznajd^2: a Community-aware Opinion Dynamics Model

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***Abstract*— It is well known that social networks are composed of many communities of nodes, where the nodes of the same community are highly connected, and few links are between the nodes of different communities. We observe scenarios in both real-world networks as well as computer networks that opinions of nodes in the networks can be aware of the existence of communities and take them into account during opinion formation. Based on the observations, we propose the first community-aware opinion dynamics model called *Sznajd*^{2} by applying the famous *Sznajd* model on the inter-community level and intra-community level. We then briefly introduce coupled fully connected networks (CFCN), analyze our model theoretically on it, and reveal that when interconnectivity parameter $v > 0.172$, nodes in the networks are surely to reach consensus on opinions along time, and when $v < 0.172$ the system might reach consensus or stay in an asymmetric stable state where some nodes disagree with others, and the state can be predicted precisely by theoretical analysis, whose correctness is also verified by simulations. For consensus performance comparison, we also perform simulations by applying our model and existing representative community-unaware models on CFCN. Simulations show that our model outperforms them, to ensure consensus on CFCN, other models require $v > 0.31$ at least, which is nearly two times big than our model.**

I. Introduction

It is well known that social networks exhibit modular structure of weakly coupled clusters, i.e., they are composed of many communities of nodes, where the nodes of the same community are highly connected, and few links are between the nodes of different communities [1]. Existing studies reveal that community structures greatly impact the dynamical evolution processes of opinions in networks [2], [3], [4], however, they assume that nodes in the networks are not aware of the existence of communities thus do not take them into account during opinion formation.

We observe that in some real world scenarios, the opinion of an individual may be influenced by each community as a whole, especially when each community is separated clearly with ground truth. For instance, in chat groups of *WeChat*, a popular mobile social network focusing on acquaintance relationships among Chinese, for the question "Is our country better than five years ago? ", considering the bias of each community because of its homogeneity, an individual might consider what his university classmates say, what his colleagues say, and what his relatives in his hometown say etc, instead of what this one says and what that one says directly. Thus even he might has dominant active relationships in the colleagues community, the way his opinion is formatted is much different than traditional opinion dynamics models which demonstrate that opinions from his colleagues will dominantly influence his own opinion.

We also observe that for consensus in P2P networks or multi-agent system, because interaction rules are determined by artificial computer softwares, opinion dynamics model can be customized for better performance [5], [6], in contrast, traditional opinion dynamics models are proposed to analyze social or physics phenomena which are objective rather than artificial. Existing studies reveal that community structures greatly decrease the opinion convergence performance, leading to slow consensus or even prevent consensus [2], [3], [4], [5]. Community-aware opinion dynamics model provides a perspective different from traditional models to deal with community structures.

Based on the observations, we propose a community-aware opinion dynamics model called *Sznajd*^{2} by applying the famous *Sznajd* model, which is one of the most influential model and successfully describes a wide variety of sociophysics situations in the past decade, on the inter-community level and intra-community level. We then briefly introduce coupled fully connected networks (CFCN), which is a topology consisting of two coupled fully connected networks, thereby mimicking the existence of communities in social networks. After that we analyze our model theoretically on CFCN, and reveal that a transition takes place at a value of the interconnectivity parameter $v \sim 0.172$. Above this value, only symmetric solutions prevail, where both communities agree with each other and reach consensus. Below this value, in contrast, the communities might reach a symmetric state or stay in an asymmetric stable state where some nodes disagree with others. Simulations on CFCN demonstrate excellent match with predictions from theoretical analysis.

To evaluate the performance of our model, we also perform simulations by applying our model and existing representative community-unaware models including *Sznajd*, *majority rule (MR)* and *voter* on CFCN. Simulations show that our model outperforms them considerably, to ensure consensus on CFCN, other models require $v > 0.31$ at least, which is nearly two times big than our model..

The rest of this paper is organized as follows. Section II describes the *Sznajd*^{2} model. Section III describes coupled fully connected networks. Section III theoretically analyzes

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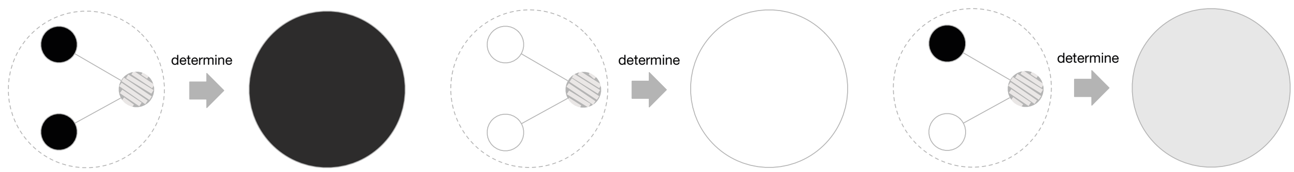
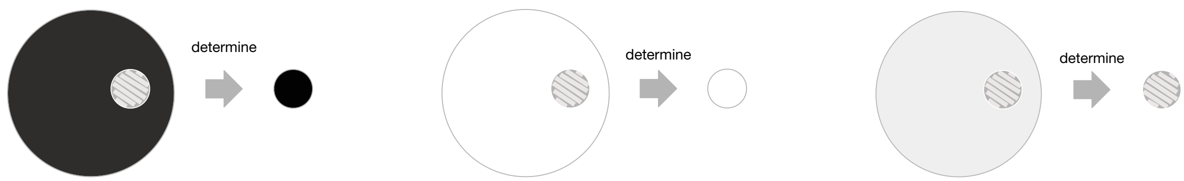


(a) Community is biased on α (b) Community is biased on β (c) Community is $unbiased$

Fig. 1. Intra-community rules. Each small circle is a node, and each big circle is a community. Black and white circle are with opinion α and β respectively, while brick circles are with any opinion, and grey circles are $unbiased$.



(a) Community is biased on α (b) Community is biased on opinion β (c) Community is $unbiased$

Fig. 2. Inter-community rules for single community. Each small circle is a node, and each big circle is a community. Black and white circle are with opinion α and β respectively, while brick circles are with any opinion, and grey circles are $unbiased$.

the model. Section V evaluates the *Sznajd*^{2} model by performing various simulations. Finally, we discuss related work in Section VI and conclude in Section VII.

II. The *Sznajd*^{2} Model

The *Sznajd*^{2} model proposed in this paper is based on the *Sznajd* model, but applied in both intra-community level and inter-community level. In the traditional *Sznajd* model for a general network, each node may has one of the two opinions α and β. At each step, a node N_k is taken at random and two nodes N_i and N_j interacts with N_k is also selected randomly. Denoting the opinion of arbitrary node N_x as s_x, if $s_i = s_j$, then set $s_k = s_i = s_j$, otherwise nothing happens. In the *Sznajd*^{2} model, each node belongs to one or more communities whose memberships are explicitly stated. Each node may has one of the two opinions α and β.

At each step, do as following first:

1) A node N_k is taken at random.
2) If N_k belongs to exact one community, then the community is selected.
3) If N_k belongs to two or more communities, then randomly pick two communities.

Then determine the biased opinion of each picked community for N_k following rules demonstrated in Fig. 1. For each community, two nodes N_i and N_j interacts with N_k is also selected randomly, In the left of this paper, the opinion of arbitrary node N_x is denoted as s_x, then:

1) If $s_i = s_j$, then the community is determined to be biased to $s_i = s_j$ as shown in Fig. 1(a) and Fig. 1(b).
2) Otherwise the community is determined to be $unbiased$ as Fig. 1(c).

If N_k belongs to exact one community, then set the opinion of N_k by rules demonstrated in Fig. 2:

1) If the community is biased to opinion $v \in \{\alpha, \beta\}$, then set $s_k = v$, as shown in Fig. 2(a) and Fig. 2(b).
2) if the community has $unbiased$ opinion, then s_k is unchanged, as shown in Fig. 2(c).

If two communities C_1 and C_2 are picked at random for N_k, then set the opinion of N_k by rules demonstrated in Fig. 3. Denoting the biased opinion of C_1 and C_2 to be v_1 and v_n respectively, the rules is described as following:

1) If $v_1 = v_2 \neq unbiased$, then set $s_k = v_1 = v_2$, as shown in Fig. 3(a) and Fig. 3(b).
2) If $v_1 = v_2 = unbiased$, then s_k is unchanged, as shown in Fig. 3(c).
3) If $v_1 = unbiased$ and $v_2 \neq unbiased$, then set $s_k = v_2$. Similarly, if $v_2 = unbiased$ and $v_1 \neq unbiased$, then set $s_k = v_1$. The rule is demonstrated in Fig. 3(d) and Fig. 3(e)
4) If $v_1 = \alpha$ and $v_2 = \beta$, or $v_1 = \beta$ and $v_2 = \alpha$, then s_k is unchanged, as demonstrated in Fig. 3(f).

III. Coupled Fully Connected Network

To analysis the *Sznajd*^{2} model accounting for community structures, and to simply the analysis we only considering the case that a network consists of exact two communities, we employ **coupled fully connected network (CFCN)** which is a generalized fully connected network, and introduced in [7]. The network consists of two fully connected communities C_1 and C_2, which are composed of n_1^T and n_2^T nodes respectively. The connection between C_1 and C_2 is ensured by interface nodes set denoted as S_0 which belongs to both C_1 and C_2.

In the following, we denote nodes that belong to only one community as **core nodes**. We also denote nodes belong to two communities as **hub nodes**. Thus core nodes set is exact

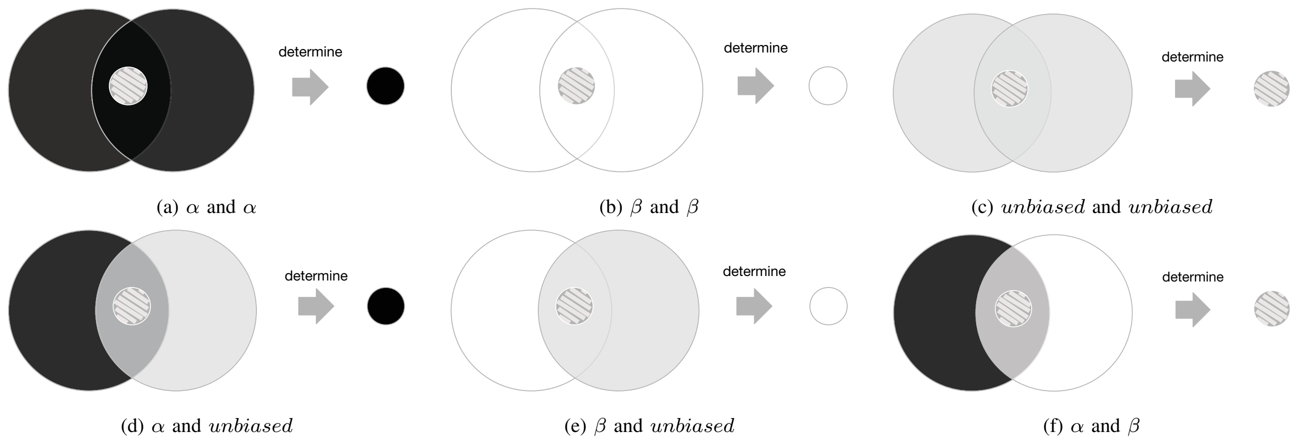


Fig. 3. Inter-community rules for two communities. Each small circle is a node, and each big circle is a community. Black and white circle are with opinion α and β respectively, while brick circles are with any opinion, and grey circles are $unbiased$.

S_0, by denoting hub nodes sets for C_1 and C_2 to be S_1 and S_2, we have the following relationships:

$$\begin{aligned} S_0 &= C_1 \bigcap C_2 \\ S_1 &= C_1 \setminus S_0 \\ S_2 &= C_2 \setminus S_0 \end{aligned} \quad (1)$$

We denote numbers of nodes for S_1, S_2 and S_0 as n_1, n_2 and n_0 respectively. By construction, those quantities satisfy:

$$\begin{aligned} n_0 + n_1 &= n_1^T \\ n_0 + n_2 &= n_2^T \end{aligned} \quad (2)$$

For the sake of clarity, we focus on equally populated communities where $n_1^T = n_2^T = n$. We also use parameter v as a measure of the interconnectivity between the communities, where $n_0 = vn$. Thus we has the following additional relationships:

$$\begin{aligned} n_1 &= n_2 = (1-v)n \\ n^T &= 2(1-v)n + vn = (2-v)n \end{aligned} \quad (3)$$

Some typical realization of CFCN can be viewed in Fig. 4, which demonstrates that greater v means tighter coupled communities. There are two limiting cases listed as following:

1) When $v = 0$, the two communities are completely disconnected, as shown in Fig. 4(a). In this case, all the nodes are core nodes, and there are no hub nodes.
2) When $v = 1$, each node in C_1 also belongs to C_2 and inversely, as shown in Fig. 4(d). In this case, all the nodes are hub nodes, and there are no core nodes, thus the network reduces to one fully connected network.

IV. Model Analysis

When $v = 1$ as presented by Fig. 4(d) , according to existing studies, the whole network asymptotically reaches consensus, i.e. all the nodes either reach opinion α or opinion β and coexistence is excluded [8]. Obviously, when $v = 0$ as presented by Fig. 4(a) , opinions in the two communities evolve independently from each other, thus the two communities reach internal consensus separately, and there is a probability $1/2$ that the opinion in C_1 is the same as in C_2, otherwise their opinions differ. However, the challenging problem is to find how the opinions evolves in the interval $v \in]0, 1[$.

In this paper, we will answer the following question: *what condition of* v *surely leads the whole network to reach consensus asymptotically*?

A. Basic Equations

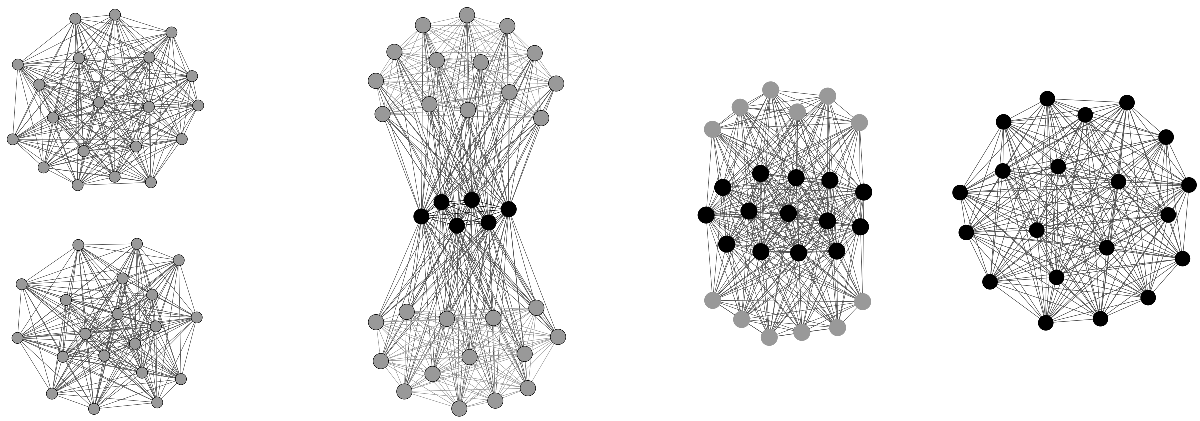
We use a_0, a_1 and a_2 to mark the densities of nodes with opinion α in the hub nodes S_0, core nodes S_1 of community C_1 and core nodes S_2 for community C_2 respectively. Similarly, b_0, b_1 and b_2 are for opinion β correspondingly. Thus we have:

$$\begin{aligned} a_0 + b_0 &= 1 \\ a_1 + b_1 &= 1 \\ a_2 + b_2 &= 1 \end{aligned} \quad (4)$$

For a node selected randomly, the probabilities that it resides in S_0, S_1 and S_2 are denoted as p_0, p_1 and p_2, according to Eq. (3) , we have the following relationships:

$$\begin{aligned} p_0 &= \frac{vn}{(2-v)n} = \frac{v}{2-v} \\ p_1 = p_2 &= \frac{(1-v)n}{(2-v)n} = \frac{1-v}{2-v} \end{aligned} \quad (5)$$

By construction, a node N_k in community C_1, i.e. either the core nodes S_1 or the hub node S_0, are connected to all the nodes in C_1, except N_k itself. The same for a node in community C_2. Considering $n_1 - 1 \sim n_1$, $n_2 - 1 \sim n_2$, $n_0 - 1 \sim n_0$ and $n - 1 \sim n$, the probabilities p_{10}, p_{11} that a randomly picked node connected to $N_k \in C_1$ resides in S_0 and S_1 respectively, as well as p_{20}, p_{22} that a randomly picked



(a) $n = 20$, $v = 0$ (b) $n = 20$, $v = 0.3$ (c) $n = 20$, $v = 0.7$ (d) $n = 20$, $v = 1$

Fig. 4. CFCN. Grey nodes are core nodes, while black nodes are hub nodes.

node connected to $N_k \in C_2$ resides in S_0 and S_2 respectively, fulfill the following equations:

$$\begin{aligned} p_{10} = p_{20} &= \frac{vn}{n} = v \\ p_{11} = p_{22} &= \frac{(1-v)n}{n} = 1 - v \end{aligned} \tag{6}$$

B. Equations for Intra-community rules

Intra-community rules are to determine the biased opinion of each picked community for a selected node N_k, following rules demonstrated in Fig. 1. We analyze the case that $N_k \in C_1$, and the case that $N_k \in C_2$ is given directly to avoid redundancy.

The case that community C_1 is determined to be biased on α stands only if the two randomly selected nodes connected with N_k in C_1 all have opinion α, as exhibited by Fig. 1(a). Obviously the probability that a randomly selected node in C_1 has opinion α is $a_0 p_{10} + a_1 p_{11}$, thus the probability of this case denoted as p_{1a} can be written by the following equation:

$$p_{1\alpha} = (a_0 p_{10} + a_1 p_{11})^2 \tag{7}$$

In the same way, the probabilities denoted as $p_{1\beta}$ and $p_{1\chi}$ for the cases that community C_1 is determined to be biased on β as illustrated by Fig. 1(b) and to be $unbiased$ as illustrated by Fig. 1(c) respectively can be written by the following equations:

$$\begin{aligned} p_{1\beta} &= (b_0 p_{10} + b_1 p_{11})^2 \\ p_{1\chi} &= 2(a_0 p_{10} + a_1 p_{11})(b_0 p_{10} + b_1 p_{11}) \end{aligned} \tag{8}$$

where $a_0 p_{10} + a_1 p_{11}$ and $b_0 p_{10} + b_1 p_{11}$ are the probabilities that a randomly selected node in C_1 has opinion α and $beta$ respectively.

Similarly, the counter parties of $p_{1\alpha}$, $p_{1\beta}$ and $p_{1\chi}$ for community C_2, denoted as $p_{2\alpha}$, $p_{2\beta}$ and $p_{2\chi}$ can be written as following:

$$\begin{aligned} p_{2\alpha} &= (a_0 p_{20} + a_2 p_{22})^2 \\ p_{2\beta} &= (b_0 p_{20} + b_2 p_{22})^2 \\ p_{2\chi} &= 2(a_0 p_{20} + a_2 p_{22})(b_0 p_{20} + b_2 p_{22}) \end{aligned} \tag{9}$$

C. Equations for Single-community Inter-community Rules

For a randomly selected node N_k belonging to core nodes set S_1 or S_2, inter-community rules for single community is applied to determine how new opinion of N_k should be set, as demonstrated in Fig. 2.

The evolution of opinions in core node set S_1 can be written as the following master equation:

$$\frac{da_1}{dt} = p_1(b_1 p_{1\alpha} - a_1 p_{1\beta}) \tag{10}$$

where $b_1 p_{1\alpha}$ corresponds to the case that the opinion of a node changes from β to α, and $a_1 p_{1\beta}$ corresponds to the reverse case.

Similarly we have the following master equation for the evolution of opinions in core node set S_2:

$$\frac{da_2}{dt} = p_1(b_2 p_{2\alpha} - a_2 p_{2\beta}) \tag{11}$$

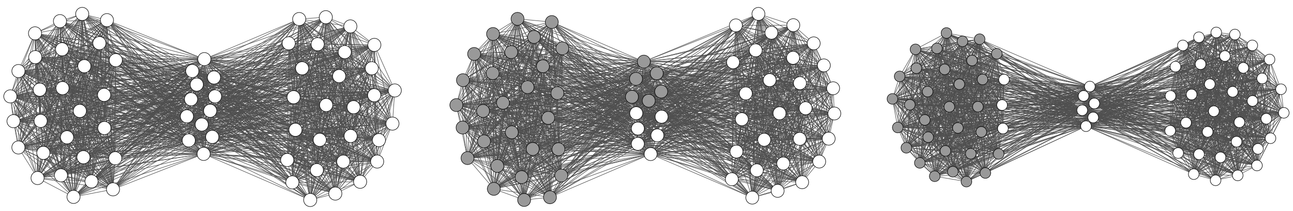
D. Equations for Two-communities Inter-community Rules

For a randomly selected node N_k belongs to hub nodes set S_0, inter-community rules for two communities is utilized to determine how new opinion of N_k should be set, as demonstrated in Fig. 3.

The evolution of opinions in hub node set S_0 can be written as the following master equation:

$$\begin{aligned} \frac{da_0}{dt} &= p_0(E - F) \\ E &= b_0(p_{1\alpha} p_{2\chi} + p_{1\chi} p_{2\alpha} + p_{1\alpha} p_{2\alpha}) \\ F &= a_0(p_{1\beta} p_{2\chi} + p_{1\chi} p_{2\beta} + p_{1\beta} p_{2\beta})) \end{aligned} \tag{12}$$

where E corresponds to the case that the opinion of a node changes from β to α, and F corresponds to the reverse case. Within E, item $p_{1\alpha} p_{2\chi}$ corresponds to the case that community C_1 is determined to be biased on α and community C_1 is $unbiased$, item $p_{1\chi} p_{2\alpha}$ corresponds to the case that community C_1 is $unbiased$ and community C_1 is determined to be biased on α and item $p_{1\alpha} p_{2\alpha}$ corresponds to the case that both of the two communities C_1 and C_2 are determined



(a) Stable symmetric state (b) Unstable asymmetric state (c) Stable asymmetric state

Fig. 5. Schematic illustration of equilibrium states

to be biased on α. The items in F correspond to the cases in the ways similar to those of E.

E. Total Contribution to the Opinion Evolution

From the previous two subsections, the total contributions to the opinion evolution in the whole network can be written by combing Eq. (10), Eq. (11) and Eq. (12), as recurred in the following:

$$\begin{aligned}\frac{da_1}{dt} &= p_1(b_1 p_{1\alpha} - a_1 p_{1\beta})\\ \frac{da_2}{dt} &= p_1(b_2 p_{2\alpha} - a_2 p_{2\beta})\\ \frac{da_0}{dt} &= p_0(b_0(p_{1\alpha}p_{2\chi} + p_{1\chi}p_{2\alpha} + p_{1\alpha}p_{2\alpha})\\ &\quad - a_0(p_{1\beta}p_{2\chi} + p_{1\chi}p_{2\beta} + p_{1\beta}p_{2\beta}))\end{aligned} \tag{13}$$

F. State Analysis

When $\frac{da_1}{dt} = \frac{da_2}{dt} = \frac{da_0}{dt} = 0$, the network is in states of **equilibrium**. According to Eq. (13), it is straightforward to show that the following states are equilibrium:

1) $a_0 = a_1 = a_2 = 0$ or $a_0 = a_1 = a_2 = 1$. In these cases all nodes in the network have the same opinion α or β, e.g. Fig. 5(a).
2) $a_0 = 0.5$, $a_1 = 1$ and $a_2 = 0$, or $a_0 = 0.5$, $a_1 = 0$, and $a_2 = 1$. In these cases, all core nodes in one community have opinion α, all core nodes in the other community have opinion β, half hub nodes in the network have opinion α, and the other half hub nodes in the network have opinion β, e.g. Fig. 5(b).

When all nodes in the network have the same opinion α or β, the network is in a **symmetric** state, where the whole network is frozen, and no change of state takes place any more along time. When both nodes with α and nodes with $beta$ coexist in the whole network, the network is in an **asymmetric** state, where fluctuations continue to take place.

When in a **unstable** state, the whole network escapes the asymmetric state in long enough times, and results in another unstable state or a stable state. When in a **stable** state, the whole network stays at the state with perhaps small deviation, even if there are continuous fluctuations.

Since a symmetric state is surely to be equilibrium and stable, the whole network **reaches consensus** only if it is in a symmetric state. If the whole network can not reach a symmetric state along time from the current state, it **fails to reach consensus**. An asymmetric state may be of equilibrium or not, and stable or unstable. It is obvious that with a stable asymmetric state, the network fails to reach consensus.

Computer simulations reveal that with a big enough value v, along time the network surely evolves to a symmetric state. In contrast, with a small enough value of v, along time the network *may* evolve to a stable asymmetric state like Fig. 5(c).

Computer simulations show that the asymmetric stable state is characterized by the forms $a_1 = 1 - \epsilon$ and $a_0 = a_2 = 0$, or $b_1 = 1-\epsilon$ and $b_0 = b_2 = 0$, where $\epsilon \in [0, 1]$, as schematically illustrated by Fig. 5(c). Because of the equivalence of the two forms in the context of this paper, we only consider the form $a_1 = 1 - \epsilon$ and $a_0 = a_2 = 0$ to avoid redundancy.

To find equilibrium solutions for this form, we should find the condition where $\frac{da_0}{dt} = \frac{da_1}{dt} = \frac{da_2}{dt}$. It is straightforward that $\frac{da_0}{dt}$ and $\frac{da_2}{dt}$ are always zero, while the equation $\frac{da_1}{dt} = 0$ leads to the following equation:

$$\frac{1}{v-2}(\epsilon - 1)(v - 1)(X\epsilon^2 - Y\epsilon + Z) = 0 \tag{14}$$

where $X = 2(v - 1)^2$, $Y = 3v^2 + 4v - 1$ and $Z = v^2$.

Obviously, $\epsilon = 1$ is an solution, which is the case corresponding to the symmetric state where all nodes in the network have the same opinion. In this case, the network is in equilibrium and stable state whatever the value of v is.

Another obvious solution is $v = 1$, which is the case that all nodes in the network are hub nodes, as illustrated in Fig. 4(d). Considering the form that $a_1 = 1 - \epsilon$ and $a_0 = a_2 = 0$, in this case all nodes in the network also have the same opinion and is stable as well.

Other solutions fulfilling $X\epsilon^2 - Y\epsilon + Z = 0$ are also possible:

$$\epsilon_{\pm} = \frac{-Y \pm \sqrt{Y^2 - 4XZ}}{2X} \tag{15}$$

which exist only if $Y^2 - 4XZ \geq 0$, thus we have the following condition:

$$(v - 1)^2(v^2 - 6v + 1) \geq 0 \tag{16}$$

Because $(v-1)^2 \geq 0$ always stands, and the only solution for $(v-1)^2 = 0$ is $v = 1$, which is already introduced above, we need only consider $v^2 - 6v + 1 \geq 0$. Thus we can easily have the condition for the existing of equilibrium solutions on the value of v as $v \leq v_c$, where $v_c = 3 - 2\sqrt{2} \sim 0.172$.

To check under what condition of v that equilibrium solutions shown in Eq. (15) are also stable states of the system, we perform stability analysis [9]. First, for an equilibrium solution, the system is linearized. Denoting the deviations of a_0, a_1 and $a2$ at equilibrium state $a_1 = 1 - \epsilon$ and $a_0 = a_2 = 0$ to be δ_0, δ_1 and δ_2 respectively, we have the following equations:

$$\begin{aligned} a_0 &= \delta_0 \\ a_1 &= 1 - \epsilon + \delta_0 \\ a_2 &= \delta_2 \end{aligned} \tag{17}$$

which is then merged into Eq. (13). Then the Lyapunov matrix M is constructed :

$$M = \begin{bmatrix} m_{00} & m_{01} & m_{02} \\ m_{10} & m_{11} & m_{12} \\ m_{20} & m_{21} & m_{22} \end{bmatrix} \tag{18}$$

where m_{ij} can be obtained by the following rule:

$$m_{ij} = \left.\frac{\partial d_i}{\partial \delta_j}\right|_{(\delta_0=0,\delta_1=0,\delta_2=0)} \tag{19}$$

Thus we have:

$$\begin{aligned} m_{00} =& -\frac{v}{v-2}(2(\epsilon-1)^2v^3 - 3(\epsilon-1)^2v^2 + \epsilon(\epsilon-2)) \\ m_{02} =& \frac{2v(\epsilon-1)^2}{v-2}(v-1)^3 \\ m_{10} =& \frac{2v(\epsilon-1)}{v-2}(v-1)((2\epsilon-1)v-2\epsilon) \\ m_{11} =& -\frac{v-1}{v-2}((6\epsilon^2-10\epsilon+4)v^2 \\ & - (12\epsilon^2-16\epsilon+4)v + 6\epsilon^2 - 6\epsilon + 1) \\ m_{22} =& -\frac{v-1}{v-2} \\ m_{01} =& m_{12} = m_{20} = m_{21} = 0 \end{aligned} \tag{20}$$

Numeric analysis is conducted for $v \in]0, v_c[$. For any equilibrium point where $\epsilon = \epsilon_+$ and $v \in]0, v_c[$, there is exact one positive eigenvalue in all the three eigenvalues of M , hence the equilibrium point is unstable. For any equilibrium point where $\epsilon = \epsilon_-$ and $v \in]0, v_c[$, all the three eigenvalues of M are negative, hence the equilibrium point is stable.

Considering the fact shown above that no asymmetric equilibrium point exists when $v >]v_c, 1[$, as well as the result of the numeric analysis on eigenvalues of M, the system exhibits a discontinuous transition at v_c:

1) When $v < v_c$, the system may reach either a symmetric or an asymmetric stable state. An asymmetric stable state fulfills $a_1 = 1 - \epsilon_-$ and $a_0 = a_2 = 0$.
2) When $v > v_c$, only the symmetric state is possible along time.

To sum up, the system is **certain** to reach consensus only if $v > v_c$.

V. Evaluations

A. Verification of Theoretical Analysis

Simulations of the *Sznajd*[2] model are performed on CFCN as shown in Fig. 6, which demonstrates excellent match with predictions from theoretical analysis. However, at the location of the transition where $v = v_c$ there is are tiny discrepancies:

1) The transition in the simulations does not appear absolute vertical. This is because that the simulations and theoretical calculation are performed for v by interval of 0.1, and $\Delta_{v=0.17} \neq \Delta_{v=0.18}$.
2) $\Delta_{v=0.18} \neq 0$. This is because that each simulation is stopped after 100 steps per node, and for $v = 0.18$, the system need more steps to reach stable state.

To sum up, simulations verifies the correctness of theoretical analysis.

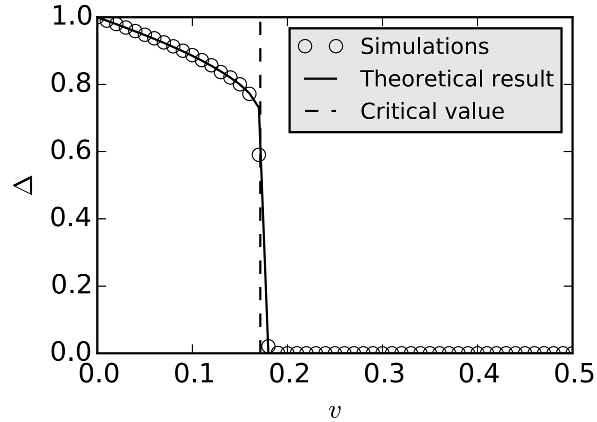


Fig. 6. Bifurcation digram of $\Delta = \|a_1 - a_2\|$ as a function of v. The vertical bar at $v = 0.172$ indicates the theoretical critical value of $v = v_c$ for the transition from asymmetric stable state to symmetric stable state. Simulations are performed on a network with $n = 4000$, started from an unstable asymmetric state with $a_0 = 0.5$, $a_1 = 1$ and $a_2 = 0$, as illustrated in Fig. 5(b), and stopped after 100 steps per node.

B. Performances Comparison on CFCN

To compare the consensus performances of the *Sznajd*[2] model to various existing representative **community-unaware** opinion dynamics models including *Sznajd*, *MR* and *voter*, simulations are also performed by apply those models on CFCN with $n = 4000$, started from an unstable asymmetric state with $a_0 = 0.5$, $a_1 = 1$ and $a_2 = 0$, as illustrated in Fig. 5(b), and stopped after 100 steps per node. Results of simulations are shown in Fig. 7 and Fig. 8 which demonstrate:

1) *Sznajd* and *MR* exhibit bifurcations similar to our model, however, the transitions are at $v_c \sim 0.31$ for *MR* and $v_c \sim 0.33$ for *Sznajd*. Thus to ensure consensus, they need much more strict community structures.
2) For *voter*, $\Delta = \|a_1 - a_2\|$ decrease to a small value quickly along with v even when $v < 0.1$, demonstrating that the densities of nodes with opinion α in community C_1 and C_2 approach almost the same quickly. However Fig. 8 shows that $\Delta = \|a - b\|$ is quite small, demonstrating that there is no dominant opinion in the network, and even $v = 0.5$, consensus can not be reached by *voter*.

To sum up, our model outperforms existing representative **community-unaware** opinion dynamics models on reaching consensus by requiring more relaxed community structures.

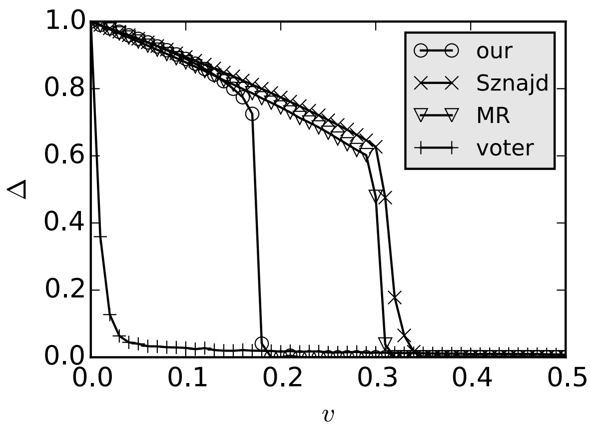


Fig. 7. $\Delta = \|a_1 - a_2\|$ as a function of v for CFCN.

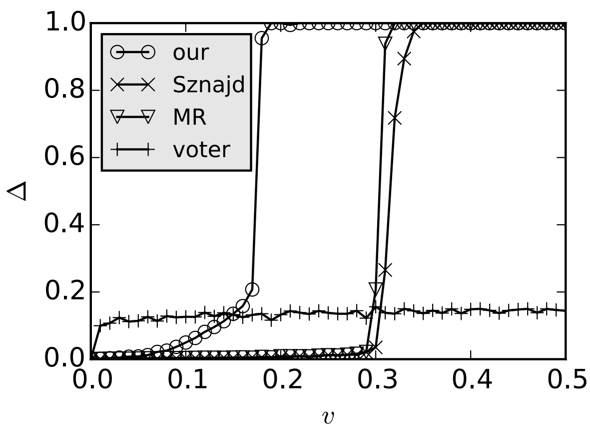


Fig. 8. $\Delta = \|a - b\|$ as a function of v for CFCN, where a and b are the densities of nodes with opinion α and β in the whole network respectively.

VI. Related Work

Opinion dynamics with communities Opinion dynamics is a field where mathematical and/or physical models and computational tools are utilized to explore the dynamical processes of the diffusion and evolution of opinions in population [10]. [11] conducted a comprehensive and influential survey on opinion dynamics from the perspective of statistics physics. [10] further gave a multidisciplinary review, concludes that opinion dynamics models follow a bottom-up modeling approach to study the aggregate dynamics of opinions, which is determined by three key features: the representation of the opinions, the local rules for the agents influence each other to change their opinions, and the overall social structure that interlinks the agents. The *MR* opinion dynamics model on a network with communities is first studied in [7], which showed that a transition takes place at a value of the interconnectivity parameter. [4] furtherly studies the *MR with probability* model on two coupled random networks in a similar approach. [2] performed simulation on *Sznajd* model in the scale-free networks with the tunable strength of community structure, and found that the smaller the community strength, the larger the slope of the exponential relaxation time distribution. [12] examine the mean consensus time of the *voter* model in the so-called two-clique graph, and showed that as the number of interclique links per node is varied, the mean consensus time experiences a crossover between a fast consensus regime and a slow consensus regime. [13] studied a nonlinear q-voter model with stochastic noise, interpreted in the social context as independence, on a duplex network, and provided evidence that even a simple rearrangement into a duplex topology may lead to significant changes in the observed behavior. However, studies mentioned above are conducted from the perspective of various existing rules on community structures, but the rules themselves does not take the community factor into consideration.

Sznajd and variations The *Sznajd* model was one of the most studied models of opinion dynamics in the last years, defined under the name USDF (united we stand, divided we fall) as a model where the society is represented by a linear chain, and people can have one of two opposite opinions [14]. The basic principle of the model is that convincing somebody is easier for two or more people than for a single individual [11]. The model has been employed to describe a wide variety of sociophysics situations in the past decade, such as marketing, finance, and politics [15]. With its wide application, it has been modified in a variety of ways [15], [11], including being adapted to a general network [8]. To examine how different types of social influence, introduced on the microscopic (individual) level, manifest on the macroscopic (society) level, [16] proposed a generalized model of opinion dynamics, that reduces to the *linear voter* model, *Sznajd* model, *q-voter* model and the *majority rule* model as special cases. However, no variation of *Sznajd* related with community structures are proposed yet.

Consensus on P2P networks Consensus is a fundamental problem for reliable distributed system to achieve agreement among distributed nodes on a value or an action [17]. Traditional consensus algorithms designed for cluster environments can not work in P2P networks whose participants numbers are unknown. To deal with this problem, graph theory based algorithms are proposed, but they are sensitive to the topology of the graph [18], [19]. Pseudo leader election based algorithms are also proposed, but they can only deal with crash failure [20]. Some other algorithms have strong assumptions on properties of the P2P networks, thus can not be applied in general networks [21]. Random walk based Byzantine consensus can tolerant topology change as well as heavy churn and achieve almost-everywhere agreement with high probability [22]. However, none of the above algorithms can survive *Sybil attack*, wherein the attacker creates a large number of pseudonymous identities, and use them to gain a disproportionately large influence [23]. Bitcoin provides Sybil-proof consensus mechanism through an ongoing chain of hash-based proof-of-work(PoW) [24], however, it can not survive attack with dominant compute power. Relationships based algorithms are considered to be more robust than other approaches against Sybil attack [25], and [5] firstly proposed a relationships based algorithm, but its performance decrease dreadfully with the presence of community structures.

VII. Conclusion

This paper observes scenarios where existing models can not describe the opinion dynamics well, or can not reach consensus with satisfying performance. Based on the observations, we propose a community-aware opinion dynamics model called *Sznajd*^{2}, and analyze it theoretically. Simulations verify the result of theoretical analysis, and demonstrate that our model outperforms existing representative community-unaware opinion dynamics models by reaching consensus considerably faster, and requiring remarkable more relaxed community structures.

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References

[1] M. Girvan and M. E. J. Newman, "Community structure in social and biological networks," *Proceedings of the National Academy of Sciences*, vol. 99, no. 12, pp. 7821–7826, 2002.

[2] W. Ru and C. Li-Ping, "Opinion Dynamics on Complex Networks with Communities," *Chinese Physics Letters*, vol. 25, p. 1502, Apr. 2008.

[3] R. Ghosh and K. Lerman, "The Impact of Network Flows on Community Formation in Models of Opinion Dynamics," *The Journal of Mathematical Sociology*, vol. 39, no. 2, pp. 109–124, 2015.

[4] R. Lambiotte and M. Ausloos, "Coexistence of opposite opinions in a network with communities," *Journal of Statistical Mechanics: Theory and Experiment*, vol. 2007, pp. P08026–P08026, Aug. 2007.

[5] H. Chen and J. Shu, "Sky: Opinion Dynamics Based Consensus for P2p Network with Trust Relationships," in *Algorithms and Architectures for Parallel Processing - 15th International Conference, ICA3PP 2015, Zhangjiajie, China, November 18-20, 2015. Proceedings, Part III*, pp. 517–531, 2015.

[6] Y. Cao, "Consensus of multi-agent systems with state constraints: a unified view of opinion dynamics and containment control," in *American Control Conference (ACC), 2015*, pp. 1439–1444, July 2015.

[7] R. Lambiotte, M. Ausloos, and J. A. Hoyst, "Majority model on a network with communities," *Physical Review E*, vol. 75, no. 3, p. 030101, 2007.

[8] F. Slanina and H. Lavicka, "Analytical results for the Sznajd model of opinion formation," *The European Physical Journal B - Condensed Matter and Complex Systems*, vol. 35, pp. 279–288, Sept. 2003.

[9] G. Nicolis, *Introduction to Nonlinear Science*. Cambridge University Press, 1995.

[10] H. Xia, H. Wang, and Z. Xuan, "Opinion Dynamics: A Multidisciplinary Review and Perspective on Future Research," *Int. J. Knowl. Syst. Sci.*, vol. 2, no. 4, pp. 72–91, 2011.

[11] C. Castellano, S. Fortunato, and V. Loreto, "Statistical physics of social dynamics," *Reviews of Modern Physics*, vol. 81, no. 2, pp. 591–646, 2009.

[12] N. Masuda, "Voter model on the two-clique graph," *Physical Review E*, vol. 90, no. 1, p. 012802, 2014.

[13] A. Chmiel and K. Sznajd-Weron, "Phase transitions in the q-voter model with noise on a duplex clique," *Physical Review E*, vol. 92, no. 5, p. 052812, 2015.

[14] K. Sznajd-Weron and J. Sznajd, "Opinion Evolution in Closed Community," *International Journal of Modern Physics C*, vol. 11, pp. 1157–1165, 2000.

[15] D. Stauffer, "Sociophysics: the Sznajd model and its applications," *Computer Physics Communications*, vol. 146, no. 1, pp. 93–98, 2002.

[16] P. Nyczka and K. Sznajd-Weron, "Anticonformity or Independence?—Insights from Statistical Physics," *Journal of Statistical Physics*, vol. 151, pp. 174–202, Feb. 2013.

[17] M. Pease, R. Shostak, and L. Lamport, "Reaching agreement in the presence of faults," *J. ACM*, vol. 27, no. 2, pp. 228–234, 1980.

[18] F. Greve and S. Tixeuil, "Knowledge connectivity vs. synchrony requirements for fault-tolerant agreement in unknown networks," in *Proceedings of the 37th Annual IEEE/IFIP International Conference on Dependable Systems and Networks*, DSN '07, (Washington, DC, USA), pp. 82–91, IEEE Computer Society, 2007.

[19] E. A. Alchieri, A. N. Bessani, J. Silva Fraga, and F. Greve, "Byzantine consensus with unknown participants," in *Proceedings of the 12th International Conference on Principles of Distributed Systems*, OPODIS '08, (Berlin, Heidelberg), pp. 22–40, Springer-Verlag, 2008.

[20] C. Delporte-Gallet, H. Fauconnier, and A. Tielmann, "Fault-tolerant consensus in unknown and anonymous networks," in *Proceedings of the 2009 29th IEEE International Conference on Distributed Computing Systems*, ICDCS '09, (Washington, DC, USA), pp. 368–375, IEEE Computer Society, 2009.

[21] H. Moniz, N. Neves, and M. Correia, "Byzantine fault-tolerant consensus in wireless ad hoc networks," *IEEE Transactions on Mobile Computing*, vol. 12, no. 12, pp. 2441–2454, 2013.

[22] J. Augustine, G. Pandurangan, and P. Robinson, "Fast Byzantine Agreement in Dynamic Networks," in *Proceedings of the 2013 ACM Symposium on Principles of Distributed Computing*, PODC '13, (New York, NY, USA), pp. 74–83, ACM, 2013.

[23] J. R. Douceur, "The sybil attack," in *Revised Papers from the First International Workshop on Peer-to-Peer Systems*, IPTPS '01, (London, UK, UK), pp. 251–260, Springer-Verlag, 2002.

[24] S. Nakamoto, "Bitcoin: A peer-to-peer electronic cash system." http://www.bitcoin.org/bitcoin.pdf, 2009.

[25] L. Alvisi, A. Clement, A. Epasto, S. Lattanzi, and A. Panconesi, "SoK: The evolution of sybil defense via social networks," in *Proceedings of the 2013 IEEE Symposium on Security and Privacy*, SP '13, (Washington, DC, USA), pp. 382–396, IEEE Computer Society, 2013.